Lecture 20

Monday, March 9, 2020 1:08 PM

- Finish proof of Lewy's Ext. Thm.

Cor1. Let $\Omega \subseteq \mathbb{C}^n$, $\zeta \in \partial\Omega$ and $\partial\Omega$ $\mathcal{C}$ $n \geq 4$) smooth near .
If Levi form of $\partial\Omega$ at ζ has both pos. and neg. eigenvalues, then $\exists\, \zeta \in \omega'$ open s.t. every $u \in \mathcal{O}(\Omega)$ extends holom. to ω'.

The Levi Problem.

Recall, we showed:

Thm A. If $\Omega \subseteq \mathbb{C}^n$ is a d.o. holom., then Ω is ψcvx.

The Levi problem is to show converse:

Thm B. If $\Omega \subseteq \mathbb{C}^n$ is ψcvx, then Ω is a d.o. holom.

This will follow from 2 results:

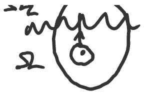
Thm 1. Let $\Omega \subseteq \mathbb{C}^n$ and assume that $\bar\partial u = f$ has solution $u \in \mathcal{C}^\infty_{(0,q)}(\Omega)$ for each $f \in \mathcal{C}^\infty_{(0,q+1)}(\Omega)$ s.t. $\bar\partial f = 0$, for each $q = 0, 1, \ldots, n-2$.
Then Ω is a d.o. holom.

Thm2. If $\Omega \subseteq \mathbb{C}^n$ is ψcvx, $f \in \mathcal{C}^\infty_{(0,q+1)}(\Omega)$ w/ $\bar\partial f = 0$, $q = 0, 1, \ldots, n-2$,
then $\exists\, u \in \mathcal{C}^\infty_{(0,q)}(\Omega)$ s.t. $\bar\partial u = f$.

Thm2 is hard part and we can only sketch this in remaining time.

Pf. of Thm1. We shall proceed by induction on n. For $n = 1$, all $\Omega \subseteq \mathbb{C}$ are d.o. holom., so nothing to prove. We assume Thm 1 holds for $\omega \in \mathbb{C}^{n-1}$! Suffices to show that for each $\zeta \in \partial\Omega$ s.t. $\exists$ open convex $D \subseteq \Omega$ w/ $\zeta \in \partial D$, $\exists\, u \in \mathcal{O}(\Omega)$ s.t. u does not extend across ζ.
(If $\exists\, \Omega_2 \not\subseteq \Omega$, $\Omega_1 \subseteq \Omega_2 \cap \Omega$ s.t. all $u \in \mathcal{O}(\Omega)$ extend into Ω_2, then $\exists$ s.t. $\zeta \in \partial\Omega \cap \Omega_2$.

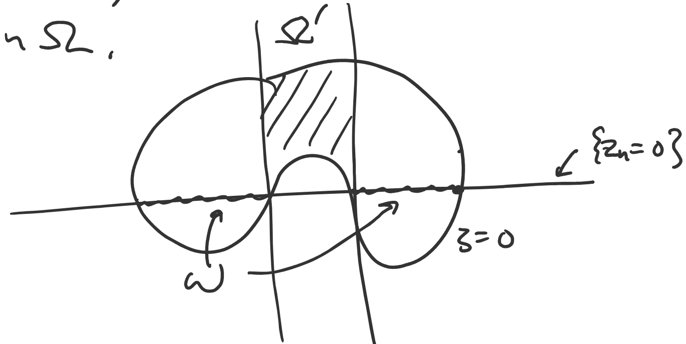
)

\ s.t. $z \in \partial\Omega \cap \Omega_2$.  /

<u>WLOG</u>: $z=0$, $D_0 = D \cap \{z_n = 0\} \neq \emptyset$. Let $\omega = \Omega \cap \{z_n=0\}$. Then $D_0 \subseteq \omega \subseteq \mathbb{C}^{n-1}$. Let $j: \mathbb{C}^{n-1} \to \mathbb{C}^n$, $\pi: \mathbb{C}^n \to \mathbb{C}^{n-1}$ be inclusion and projection: $j(z_1,\dots,z_{n-1}) = (z_1,\dots,z_{n-1},0)$; $\pi(z_1,\dots,z_n) = (z_1,\dots,z_{n-1})$.

<u>Claim</u>: If $f \in \mathcal{C}^\infty_{(0,q)}(\omega)$, $\bar\partial f = 0$, $q = 0,1,\dots,n-2$, then $\exists\, F \in \mathcal{C}^\infty_{(0,q)}(\Omega)$ s.t. $j^*F = f$.

<u>Pf of claim</u>. Let $\Omega' \subseteq \Omega$ be $\{z \in \Omega : \pi(z) \notin \omega\}$ $= \pi^{-1}(\mathbb{C}^{n-1} \setminus \omega)$. Note that ω $(\subseteq \mathbb{C}^{n-1} \subseteq \mathbb{C}^n)$ and Ω' are disjoint and closed in Ω.



We can find (not obvious, but true) $\varphi \in \mathcal{C}^\infty(\Omega)$ s.t. $\varphi = 0$ on open nbhd of Ω' and $\varphi = 1$ on open nbhd of ω. Let $f \in \mathcal{C}^\infty_{(0,q)}(\omega)$, $\bar\partial f = 0$. Consider $F = \varphi \pi^* f + z_n v$, then $j^*F = f$. We need to find v s.t. $\bar\partial F = 0 \iff \bar\partial v = \frac{1}{z_n}(\bar\partial\varphi \wedge \pi^* f)$ since $\varphi\bar\partial\pi^*f = 0$. Moreover,

$\frac{1}{z_n}(\bar\partial\varphi \wedge \pi^* f) \in \mathcal{C}^\infty_{(0,q+1)}(\Omega)$ since $\bar\partial\varphi = 0$ in open nbhd of $\omega = \Omega \cap \{z_n = 0\}$. Also, $\bar\partial\left(\frac{1}{z_n}(\bar\partial\varphi \wedge \pi^* f)\right) = 0$. By assumption $\exists\, v \in \mathcal{C}^\infty_{(0,q)}$ s.t. $\bar\partial v = \frac{1}{z_n}(\bar\partial\varphi \wedge \pi^* f) \Rightarrow \bar\partial F = 0$. ■

We now complete the pf of Thm 1. Since $D_0 \subseteq \omega$ and $z \in \partial D_0$, we have $z \in \partial\omega$. We now claim that ω is d.o.holom. By the inductive hypothesis, it suffices to verify that $\bar\partial u = f$ is solvable in $\mathcal{C}^\infty_{(0,q)}(\omega)$ for all $f \in \mathcal{C}^\infty_{(0,q+1)}(\omega)$ w/ $\bar\partial f = 0$ and $q = 0,1,\dots,n-3$. By claim above, any such f can be

It suffices to verify that $\bar\partial u = f$ is solvable in $\mathcal{C}^\infty_{(0,q)}(\omega)$ for all $f \in \mathcal{C}^\infty_{(0,q+1)}(\omega)$ w/ $\bar\partial f = 0$ and $q = 0, 1, \ldots, n-3$. By claim above, any such f can be extended to $F \in \mathcal{C}^\infty_{(0,q+1)}(\Omega)$ w/ $\bar\partial F = 0$. By assumption in Thm 1, $\exists\, v \in \mathcal{C}^\infty_{(0,q)}(\Omega)$ w/ $\bar\partial v = F$. Thus, $u = j^* v$ solves $\bar\partial u = f$ in ω. By inductive hypothesis, ω is a d.o. holom. Since $\zeta \in \partial\omega$ $\exists\, u \in \mathcal{O}(\omega)$ s.t. u cannot be extended across ζ. By claim again, we can extend u to $U \in \mathcal{O}(\Omega)$ s.t. $j^* U = U|_{\omega} = u$. Then U cannot be extended across ζ, which completes the pf. $\boxtimes$